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STRESS AND STRAIN DISTRIBUTIONS IN A THICK-WALLED CYLINDER OF STRAIN-HARDENING MATERIAL, ELASTIC-PLASTICALLY STRAINED BY INTERNAL PRESSURE

by C. K. Liu

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DEFINITION OF SYMBOLS

Symbol	
$S/2$	Yield point in Shear
σ_r, σ_θ, σ_z	Radial, circumferential, axial stresses
ϵ_r, ϵ_θ, ϵ_z	Radial, circumferential, axial strains
Υ	A function used in Hencky's theory
ξ	Υ/H
$3H$	Bulk modulus of elasticity (corresponding to 3α in Ref. 4
E	Young's modulus of elasticity
G	Modulus of rigidity
ν	Poisson's ratio
m	A numerical constant denoting the exponent of a simple power function
r	(variable) radius at a point of the cylinder
r_o	Inner radius of the cylinder
kr_o	Outer radius of the cylinder
nr_o	Radius of elastic-plastic interface
w	r/r_o
f	A function used in the modified criterion of yielding
A, B, C	Constants of integration

Other symbols will be defined as they appear in the text.

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SUMMARY

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An attempt is made to extend Allen and Sopwith's solution for an elastic-plastic thick-walled cylinder of ideally plastic material to strain-hardening material. Tresca's criterion of yielding is modified so that the effective stress (usually considered a function of the effective strain) is proportional to a simple power function of the radius in the plastic zone. An analysis of Allen and Sopwith's solution for ideally plastic material seems to support and justify the proposed modification of the yielding criterion. The exponent of the power function is taken as -3/2, -5/4, -1, and -1/2, in each case yielding an analytic solution corresponding to a particular strain-hardening material. Examples worked are for the case of plane strain and for a ratio of outer to inner radii of 2.0. When the exponent is taken to be zero, the solution becomes identical to that of Allen and Sopwith. For exponents other than the five mentioned above, numerical or graphical methods of integration may be used.

Author

INTRODUCTION

During the past two decades many solutions have been presented for the problem of plastically strained thick-walled cylinders. Because of different philosophies and purposes of the originators, some solutions emphasize the practical aspects of the problem (REF. 1, 2, 3); others are elegant in their mathematical operations (REF. 4, 5, 6); some are analytical (REF. 5, 6), and others numerical (REF. 7, 8, 9).

For engineering applications and for mathematical expediency, the authors impose various end conditions on the cylinder, or they

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hypothesize that the material is incompressible. In some cases, where plastic yielding is predominant, the material is considered as being plastic-rigid. Moreover, the solutions also differ as the material is assumed to strain-harden or to be ideally plastic.

Steele (REF. 6) presented a solution to this problem in closed and simple form. It is based on Hencky's total plastic strain theory, Tresca's yielding criterion, and the assumption that the material is incompressible in the elastic state. In assuming this, Steele postulates that across the entire thickness of the cylinder (in both the elastic and plastic zones) the strain decreases in inverse proportion to a function of the square of the radius at which the strain is sought.

Earlier, Allen and Sopwith (REF. 4) effected an analytical solution to the problem. Since they include the compressibility of the material in the elastic zone, their solution is more rigorous mathematically, physically sound, and reasonably convenient to apply. Yet, it does not have provisions to take into account the strain-hardening property of the material. The solution in this report extends Allen and Sopwith's solution to strain-hardening material by using a modified Tresca criterion of yielding.

As shown by Hill, Lee and Tupper (REF. 7), the more exact theory governing the stress-strain relationship in the plastic region is the so-called "incremental theory" of which Hencky's total strain theory is a special case. The latter however, is the form used in the vast majority of work on plasticity (REF. 4). Furthermore, the solution obtained by applying Hencky's theory is a close approximation to that obtained by the more exact theory (REF. 9). It is probable that these authors (REF. 2, 3, 5, 9), as well as Allen and Sopwith, have used Hencky's theory for no other reason than mathematical simplicity.

METHOD OF SOLUTION

The compressibility of the material can be expressed as follows:

$$\epsilon_r + \epsilon_\theta + \epsilon_z = 3H\,(\sigma_r + \sigma_\theta + \sigma_z) \tag{1}$$

where $H = \dfrac{1 - 2\nu}{3E}$

By Hencky's total strain theory, the following relationship between the plastic strain and stress is postulated:

$$\frac{\epsilon_\theta - \epsilon_r}{\sigma_\theta - \sigma_r} = \frac{\epsilon_r - \epsilon_z}{\sigma_r - \sigma_z} = \frac{\epsilon_z - \epsilon_\theta}{\sigma_z - \sigma_\theta} = 3\gamma \tag{2}$$

Tresca's yielding criterion is modified here as follows:

$$\sigma_\theta - \sigma_r = S\,f \tag{3}$$

where f is a certain function dependent on the strain-hardening property of the material. This function (f) must assume a value of unity when the strain-hardening effect of the material is absent, or when plastic yielding is impending.

Continuity at elastic-plastic interface, in addition to the given conditions at the inner and outer cylinder surfaces, provides the necessary equations from which the stresses may be determined. These conditions are:

$$\left.\begin{aligned} \sigma_r\Big|_{\text{elastic}} &= \sigma_r\Big|_{\text{plastic}} && \text{at } w = n \\ \sigma_\theta - \sigma_r &= S && \text{at } w = n \\ \sigma_r &= O && \text{at } w = k \\ \sigma_r &= -p && \text{at } w = 1 \end{aligned}\right\} \tag{4}$$

The stress distribution in the elastic zone is given by the Lame solution, that is,

$$\sigma_r = A - \frac{B}{w^2} \tag{5}$$

$$\sigma_\theta = A + \frac{B}{w^2} \tag{6}$$

In the plastic zone ($1 \leqq w \leqq n$), the stresses are found from the equation of equilibrium

$$w \frac{d \sigma_r}{d w} = \sigma_\theta - \sigma_r = S f \tag{7}$$

hence,

$$\sigma_r = S \int \frac{f}{w} dw + C \tag{8}$$

$$\sigma_\theta = S \left[f + \int \frac{f}{w} dw \right] + C \tag{9}$$

Applying boundary conditions (4) the constants A, B, and C are found to be

$$A = \frac{S}{2} \frac{n^2}{k^2} \tag{10}$$

$$B = \frac{S}{2} n^2 \tag{11}$$

$$C = S \left(\frac{n^2 - k^2}{2k^2} \right) - S \int \frac{f}{w} dw \Bigg|_{w = n} \tag{12}$$

Hence, the stresses across the thickness of the cylinder (FIG 1), are

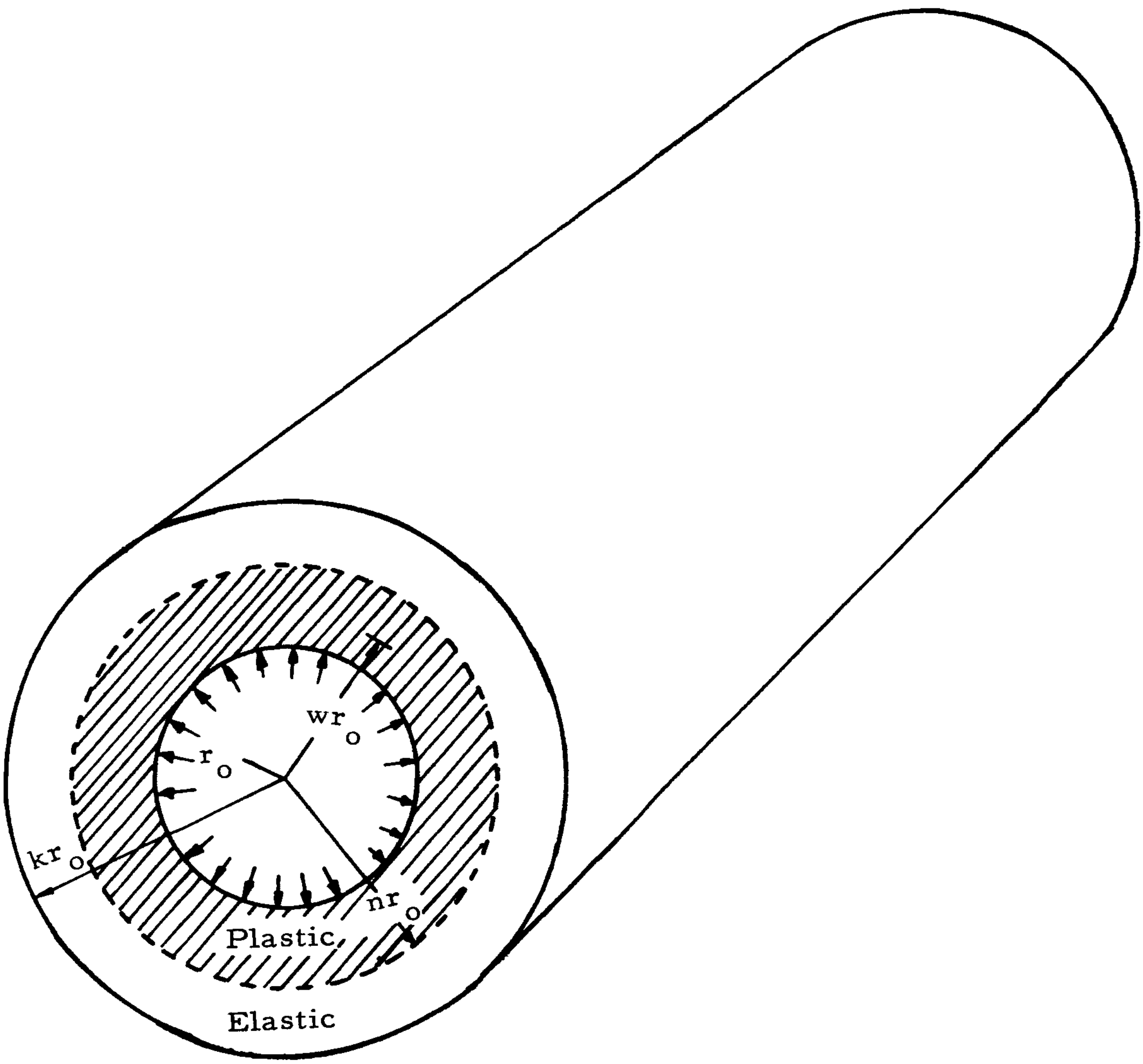

FIGURE 1. AN INFINITE HOLLOW CYLINDER ELASTIC-PLASTICALLY STRAINED BY INTERNAL PRESSURE

$$\frac{\sigma_r}{S} = \frac{n^2}{2}\left[\frac{1}{k^2} - \frac{1}{w^2}\right]$$

$$\frac{\sigma_\theta}{S} = \frac{n^2}{2}\left[\frac{1}{k^2} + \frac{1}{w^2}\right] \qquad \text{elastic zone } (n \leqq w \leqq k) \quad (13)$$

$$\frac{\sigma_z}{S} = \frac{E\epsilon_z}{S} + \nu\frac{n^2}{k^2}$$

$$\frac{\sigma_r}{S} = \int_n^w \frac{f}{w}\,dw + \frac{n^2 - k^2}{2k^2}$$

$$\frac{\sigma_\theta}{S} = f + \int_n^w \frac{f}{w} + \frac{n^2 - k^2}{2k^2} \qquad \text{plastic zone } (1 \leqq w \leqq n) \quad (14)$$

$$\frac{\sigma_z}{S} = \text{to be determined later}$$

The internal pressure (p) can be found from condition (4) as

$$\frac{p}{S} = \int_n^1 \frac{f}{w}\,dw - \frac{k^2 - n^2}{2k^2} \qquad (15)$$

From the condition of compressibility (1) and Hencky's total strain theory (2), the strain in the plastic zone can be expressed in terms of the function Υ and the stresses given by equations (13) and (14).

$$\epsilon_z = (H + 2\Upsilon)\sigma_z + (H - \Upsilon)(\sigma_r + \sigma_\theta) \qquad (16)$$

$$\epsilon_r = (H + 2\Upsilon)\sigma_r + (H - \Upsilon)(\sigma_z + \sigma_\theta) \qquad (17)$$

$$\epsilon_\theta = (H + 2\Upsilon)\sigma_\theta + (H - \Upsilon)(\sigma_r + \sigma_z) \qquad (18)$$

Rewriting, equation (16) becomes

$$\sigma_z = \frac{\epsilon_z}{H + 2\Upsilon} - \frac{H - \Upsilon}{H + 2\Upsilon} \cdot (\sigma_r + \sigma_\theta) \tag{19}$$

Substitution of equation (14) into (19) gives

$$\sigma_z = \frac{\epsilon_z}{H + 2\Upsilon} - \frac{H - \Upsilon}{H + 2\Upsilon} \left[f + 2 \int_n^w \frac{f}{w} \, dw + \frac{n^2 - k^2}{k^2} \right] S \tag{20}$$

Consequently, the three components of strain are

$$\epsilon_\theta = \frac{H - \Upsilon}{H + 2\Upsilon} \epsilon_z + S \left[\frac{3\Upsilon(2H + \Upsilon)}{H + 2\Upsilon} f + \frac{9H\Upsilon}{H + 2\Upsilon} \left(\int_n^w \frac{f}{w} \, dw + \frac{n^2 - k^2}{2k^2} \right) \right] \tag{21}$$

$$\epsilon_r = \frac{H - \Upsilon}{H + 2\Upsilon} \epsilon_z + S \left[\frac{3\Upsilon(H - \Upsilon)}{H + 2\Upsilon} f + \frac{9H\Upsilon}{H + 2\Upsilon} \left(\int_n^w \frac{f}{w} \, dw + \frac{n^2 - k^2}{2k^2} \right) \right] \tag{22}$$

ϵ_z = constant, to be determined from the end condition of the cylinder (23)

To further investigate the relationship among the strains and the function f, it is necessary to apply the equation of compatibility

$$w \frac{d\epsilon_\theta}{dw} = \epsilon_r - \epsilon_\theta \tag{24}$$

The following equation is obtained after equations (21) and (22) are substituted into equation (24)

$$(H + 2\Upsilon)(2H + \Upsilon) \left(2 \frac{dw}{w} + \frac{df}{f} \right) = \frac{d\Upsilon}{\Upsilon} \left[\frac{1}{f} \frac{H \epsilon_z}{S} - 3H^2 \left(\int_n^w \frac{f}{w} \, dw + \frac{n^2 - k^2}{2k^2} \right) - 2(H^2 + H\Upsilon + \Upsilon^2) \right] \tag{25}$$

Now, if $f = 1$ in equation (25) and we write

$$\Upsilon = \frac{x - 1}{2} H, \quad \frac{y}{3} = 2 \ln(w/n) + \frac{n^2}{k^2} - \frac{2}{3} \frac{\epsilon_Z}{SH} \tag{26}$$

it is found that equation (25) reduces to

$$\frac{dy}{dx} + \frac{3y}{x(x - 1)(x + 3)} = - \frac{3x}{(x - 1)(x + 3)} \tag{27}$$

Equation (27) is identical to that of Allen and Sopwith (REF. 4, p. 70) for an ideally plastic material. Its solution is found as

$$\ln(w/n) = \frac{\epsilon_Z}{3\,HS} - \frac{n^2}{2k^2} + \frac{1 + 3u^4}{4u^3} \left\{ \frac{2}{3} \left(\frac{3n^2}{k^2} - \frac{2\epsilon_Z}{SH} \right) \frac{\left(\frac{1}{1 + 12GH}\right)^{3/4}}{1 + 3\left(\frac{1}{1 + 12GH}\right)} \right.$$

$$+ \left(\tan^{-1} u - \tanh^{-1} u \right) - \left[\tan^{-1}\left(\frac{1}{1 + 12GH}\right)^{1/4} \right.$$

$$\left. \left. - \tanh^{-1}\left(\frac{1}{1 + 12GH}\right)^{1/4} \right] \right\} \tag{28}$$

where

$$u = \left(\frac{\Upsilon}{\Upsilon - 2H}\right)^{1/4} \qquad GH = \frac{1 - 2\nu}{6(1 + \nu)}$$

For convenience in later computations, it is assumed that $\epsilon_Z = 0$ (that is, the cylinder has clamped ends) and that $\nu = 1/3$. Thus, equation (28) is simplified greatly:

$$\ln(w/n) = -\frac{n^2}{2k^2} + \frac{1-3u^4}{4u^3}\left\{\left(\frac{2}{3}\right)^{7/4}\frac{n^2}{k^2} - \left[\tan^{-1}(2/3)^{1/4} - \tanh^{-1}(2/3)^{1/4}\right] + (\tan^{-1}u - \tanh^{-1}u)\right\} \tag{29}$$

Equation (29) correlates the quantities w/n and u (or γ, in turn) in a transcendental manner. The term n/k appears to be a parameter that may or may not affect the (w/n) versus γ relationship appreciably. To determine this, w/n is plotted against γ/H ($= \zeta$) in FIG 2 for a series of values of n/k : 1.0, 0.875, 0.750, 0.625, 0.500. In the actual cylinder, w lies between 1 and n in the plastic zone. The corresponding range of w/n is $1/n$ to 1. Hence the plot of γ versus w/n for a particular value of n/k has physical significance only when w/n is within the range $[1/\ k(n/k)]$ to 1. This is the way to determine the range of interest of w/n for any choice of k.

In this report the case of $k = 2$ is considered. It is found that while n/k varies from 1 to 0.5, the maximum range of variation in w/n never exceeds 0.37 percent. With the scale of plotting used in FIG 2, it is difficult to distinguish these four curves, one overlapping whose relationship with γ is not appreciably affected by a change in the value of n for a given value of k.

Considering w/n as an independent variable, it may be advantageous to assume that the function f is a function of w/n only. A possible form of f may be written as

$$f = C_1(w/n)^m + C_2(w/n)^\alpha + C_3(w/n)^\beta + \ldots\ldots\ (1 \leq w \leq n) \tag{30}$$

where C_1, C_2, $C_3, \ldots$; m, α, β, ... are constants. The purpose of shaping f in this manner is to simulate the unknown relationship between w/n and the respective plastic strain. Here, according to the characteristics of f, it will be necessary that when $w/n = 1$

$$C_1 + C_2 + C_3 + \ldots\ldots\ldots\ldots = 1 \tag{31}$$

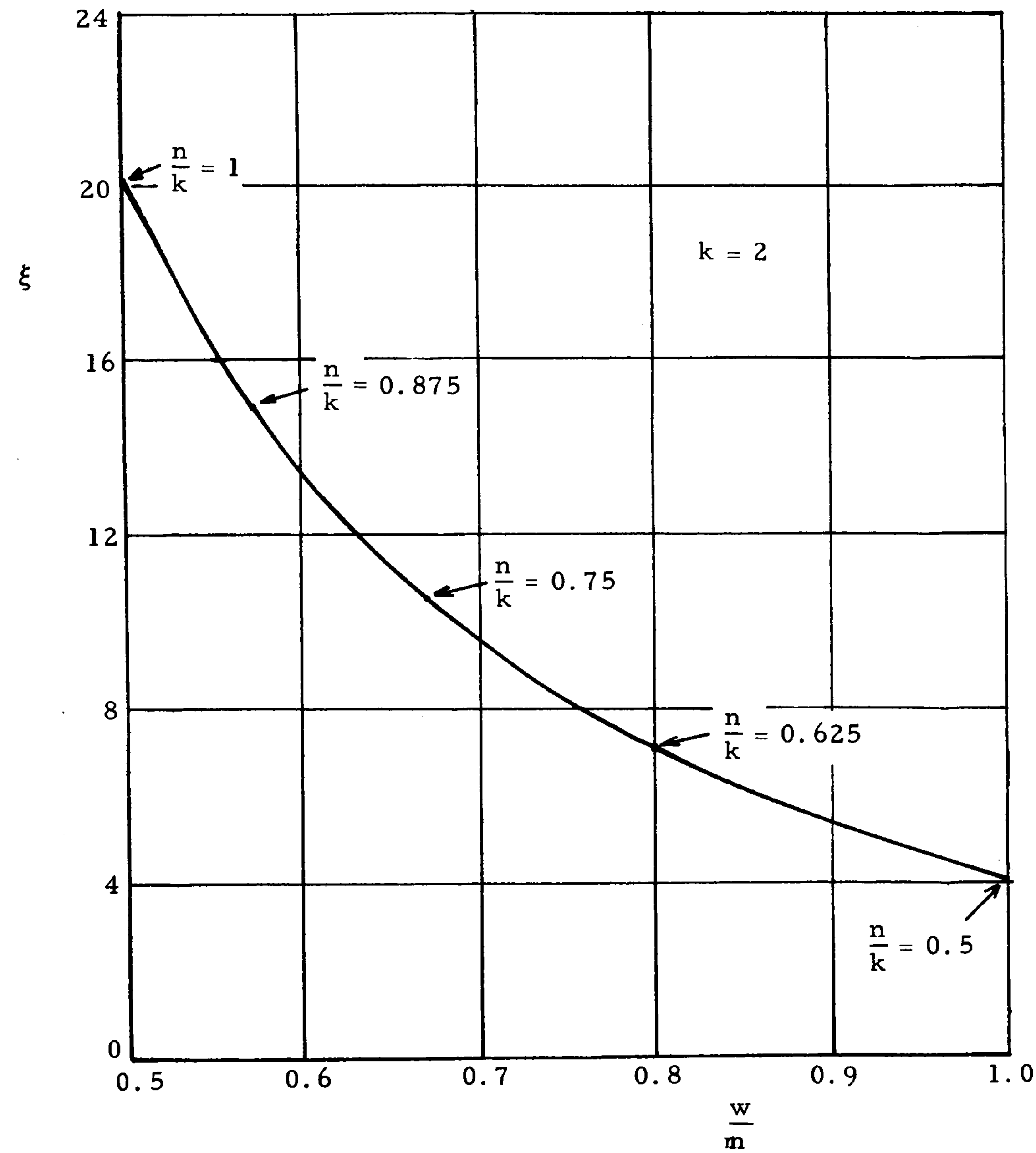

FIGURE 2. A DIMENSIONLESS PLOT SHOWING RELATIONSHIP BETWEEN ξ AND $\frac{w}{n}$ FOR VARIOUS VALUES OF $\frac{n}{k}$

Let it be further assumed that the constant C_1 is predominantly larger than the sum of the rest of the terms, then $C_1 = 1$, $C_2 = C_3 = \ldots. = 0$ and

$$f = (w/n)^m \tag{32}$$

With the function f expressed by equation (32), equation (25) becomes

$$\frac{df}{dY} + \frac{(3 + 2m)\ H^2 + 2mHY + 2mY^2}{(2 + m)\ Y\ (H + 2Y)\ (2H + 2Y)}\ f$$

$$= \frac{m}{m + 2}\ \frac{H}{Y(2H + Y)\ (H + 2Y)} \left[\frac{\epsilon_z}{S} - 3H\left(\frac{n^2 - k^2}{2k^2} - \frac{1}{m}\right)\right] \tag{33}$$

The problem is thus reduced to solving the first order, ordinary differential equation (33). The solution of equation (33) is

$$\left(\frac{w}{n}\right)^m = \frac{1 + 2\xi}{\xi^{\frac{2m + 3}{2(m + 2)}}\ (2 + \xi)^{\frac{2m + 1}{2(m + 2)}}} \left\{ \frac{m}{m + 2} \left[\frac{\epsilon_z}{SH} + \frac{3}{2}\left(\frac{m + 2}{m} - \frac{n^2}{k^2}\right)\right] \int \frac{d\xi}{\xi^{\frac{1}{2(m + 2)}}\ (1 + 2\xi)^2 (2 + \xi)^{\frac{3}{2(m + 2)}}} + K' \right\} \tag{34}$$

in which $\xi = \gamma/H$, and K', the constant of integration, is determined by the condition: $w = n$, or $f = 1$, $\xi = 1/(6GH)$. Thus:

$$K' = \frac{\left(\frac{1}{6GH}\right)^{\frac{2m+3}{2(M+2)}} \left(2 + \frac{1}{6GH}\right)^{\frac{2m+1}{2(m+2)}}}{1 + \frac{1}{3GH}} - \frac{m}{m+2}$$

$$+ \left[\frac{\epsilon_z}{SH} + \frac{3}{2}\left(\frac{m+2}{m} - \frac{n^2}{k^2}\right)\right] \bullet$$

$$\bullet \int \frac{d\xi}{\xi^{\frac{1}{2(m+2)}} (1 + 2\xi)^2 (2 + \xi)^{\frac{3}{2(m+2)}}} \Bigg|_{\xi = \frac{1}{6GH}} \tag{35}$$

Consequently. equation (34) becomes

$$\left(\frac{w}{m}\right)^m = \frac{\dfrac{1 + 2\xi}{\xi^{\frac{2m+3}{2(m+2)}} (2 + \xi)^{\frac{2m+1}{2(m+2)}}}}{\dfrac{1 + \frac{1}{3GH}}{\left(\frac{1}{6GH}\right)^{\frac{2m+3}{2(m+2)}} \left(2 + \frac{1}{6GH}\right)^{\frac{2m+1}{2(m+2)}}}} = \frac{m}{m+2}\left[\frac{\epsilon_z}{SH} + \frac{3}{2}\left(\frac{m+2}{m} - \frac{n^2}{k^2}\right)\right] \bullet$$

$$\bullet \frac{1 + 2\xi}{\xi^{\frac{2m+3}{2(m+2)}} (2 + \xi)^{\frac{2m+1}{2(m+2)}}}$$

$$\int_{\frac{1}{6GH}}^{\xi} \frac{d\xi'}{\xi^{\frac{1}{2(m+2)}} (1 + 2\xi')^2 (2 + \xi')^{\frac{3}{2(m+2)}}} \tag{36}$$

The evaluation of the integral in equation (36) for the general case is quite cumbersome, but the integration can be carried out easily if the exponent m is given particular values, such as -3/2, -5/4, -1, -1/2, etc. Four solutions corresponding to these four exponents are given below.

$m = -3/2$,

$$\left(\frac{w}{n}\right)^{-3/2} = \frac{(1+2\xi)(2+\xi)^2}{\left(1+\frac{1}{3GH}\right)\left(2+\frac{1}{6GH}\right)} - 3\left[\frac{\epsilon_z}{SH} - \frac{3}{2}\left(\frac{1}{3}+\frac{n^2}{k^2}\right)\right](1+2\xi)(2+\xi)^2 .$$

$$\left[\ln\frac{\xi'}{\xi'+2} + \frac{22}{27}\frac{1}{(2+\xi')} + \frac{2}{9}\frac{1}{(2+\xi')^2} + \frac{64}{27}\frac{1}{(1+2\xi')}\right]_{\frac{1}{6GH}}^{\xi} \tag{37}$$

$m = -5/4$,

$$\left(\frac{w}{n}\right)^{-5/4} = \frac{\dfrac{(1+2\xi)(2+\xi)}{\xi^{1/3}}}{\dfrac{\left(1+\frac{1}{3GH}\right)\left(2+\frac{1}{6GH}\right)}{\left(\frac{1}{6GH}\right)^{1/3}}} - \frac{5}{3}\left[\frac{\epsilon_z}{SH} - \frac{3}{2}\left(\frac{3}{5}+\frac{n^2}{k^2}\right)\right]\frac{(1+2\xi)(2+\xi)}{\xi'^{1/3}}$$

$$\left[\frac{4}{9}\frac{\xi'^{1/3}}{1+2\xi'} + \frac{1}{18}\frac{\xi'^{1/3}}{2+\xi'}\right.$$

$$+ \frac{5}{27(2)^{2/3}}\ln\frac{(2)^{1/3}+\xi'^{1/3}}{\left\{\xi'^{2/3}-(2)^{1/3}\xi'^{1/3}+(2)^{2/3}\right\}^{1/2}}$$

$$\left. + \frac{5(3)^{1/2}}{27(2)^{2/3}}\tan^{-1}\frac{2\xi'^{1/3}-(2)^{1/3}}{(3)^{1/3}(2)^{1/3}}\right]_{\frac{1}{6GH}}^{\xi} \tag{38}$$

$m = -1,$

$$\left(\frac{w}{n}\right)^{-1} = \frac{\dfrac{(1 + 2\xi)(2 + \xi)^{1/2}}{\xi^{1/2}}}{\dfrac{\left(1 + \frac{1}{3GH}\right)\left(2 + \frac{1}{6GH}\right)^{1/2}}{\left(\frac{1}{6GH}\right)^{1/2}}} - \left[\frac{\epsilon_z}{SH} - \frac{3}{2}\left(1 + \frac{n^2}{k^2}\right)\right] \frac{(1 + 2\xi)(2 + \xi)^{1/2}}{\xi^{1/2}} \bullet$$

$$\bullet \frac{2}{9}\left(\frac{\xi' - 2}{\xi'^2 - \xi' + 1} - \frac{1}{\xi' + 1}\right) \Bigg| \begin{matrix} 1 + \xi + [\xi(\xi + 2)]^{1/2} \\ \dfrac{1 + 6GH + (1 + 12GH)^{1/2}}{6GH} \end{matrix} \tag{39}$$

$m = -1/2,$

$$\left(\frac{w}{n}\right)^{-1/2} = \frac{\dfrac{1 + 2\xi}{\xi^{2/3}}}{\dfrac{1 + \frac{1}{3GH}}{\left(\frac{1}{6GH}\right)^{2/3}}} - \frac{1}{3}\left[\frac{\epsilon_z}{SH} - \frac{3}{2}\left(3 + \frac{n^2}{k^2}\right)\right] \frac{1 + 2\xi}{\xi^{2/3}} \bullet$$

$$\bullet \left[\frac{2}{3}\,\frac{\xi'^{2/3}}{1 + 2\xi'} + \frac{1}{9(2)^{1/3}} \ln \frac{\left\{\xi'^{2/3} - (2)^{1/3}\xi'^{1/3} + (2)^{2/3}\right\}^{1/2}}{(2)^{1/3} + \xi'^{1/3}}\right.$$

$$\left. + \frac{(3)^{1/2}}{9(2)^{1/3}} \tan^{-1} \frac{2\xi'^{1/3} - (2)^{1/3}}{(2)^{1/3}(3)^{1/2}}\right]_{\frac{1}{6GH}}^{\xi} \tag{40}$$

Again, assuming $\epsilon_z = 0$ for a cylinder with clamped ends, and $\nu = 1/3$ ($6GH = 1/4$) it is possible to examine the relationship between w/n and ξ for various values of m. In FIG 3, four curves are plotted so that the effect of m on ξ can be brought out. It is seen that the material with $m = -3/2$ at a fixed value of w/n yields the smallest value of ξ among the four materials. Note that fixed value of n corresponds to a fixed magnitude of the internal pressure (p) through equation (15). In FIG 2, a dimensionless plot is shown for the relationship between ξ and w/n for various values of n/k. The single curve shown is actually four curves of different lengths overlapping one another.

With the interdependence between $(w/n)^m$ and ξ determined, it is possible to check back to see what degree of strain-hardening each value of m represents. From the yielding criterion (3) it is assumed

$$\frac{\sigma_\theta - \sigma_r}{S} = f \tag{3}$$

and from Hencky's total strain theory (2),

$$\frac{\epsilon_\theta - \epsilon_r}{\sigma_\theta - \sigma_r} = 3Y \tag{2}$$

Combining equations (3) and (2),

$$\frac{\epsilon_\theta - \epsilon_r}{S\,H} = 3\frac{Y}{H}f = 3\xi f \tag{41}$$

For certain values of m, ξ and f are related to each other according to the curves in FIG 3. Hence, by plotting f against $3\xi f$, a dimensionless stress-strain diagram of this particular material can be obtained. Four curves of this kind, including that for an ideally plastic material (m = 0, are shown in FIG 4.

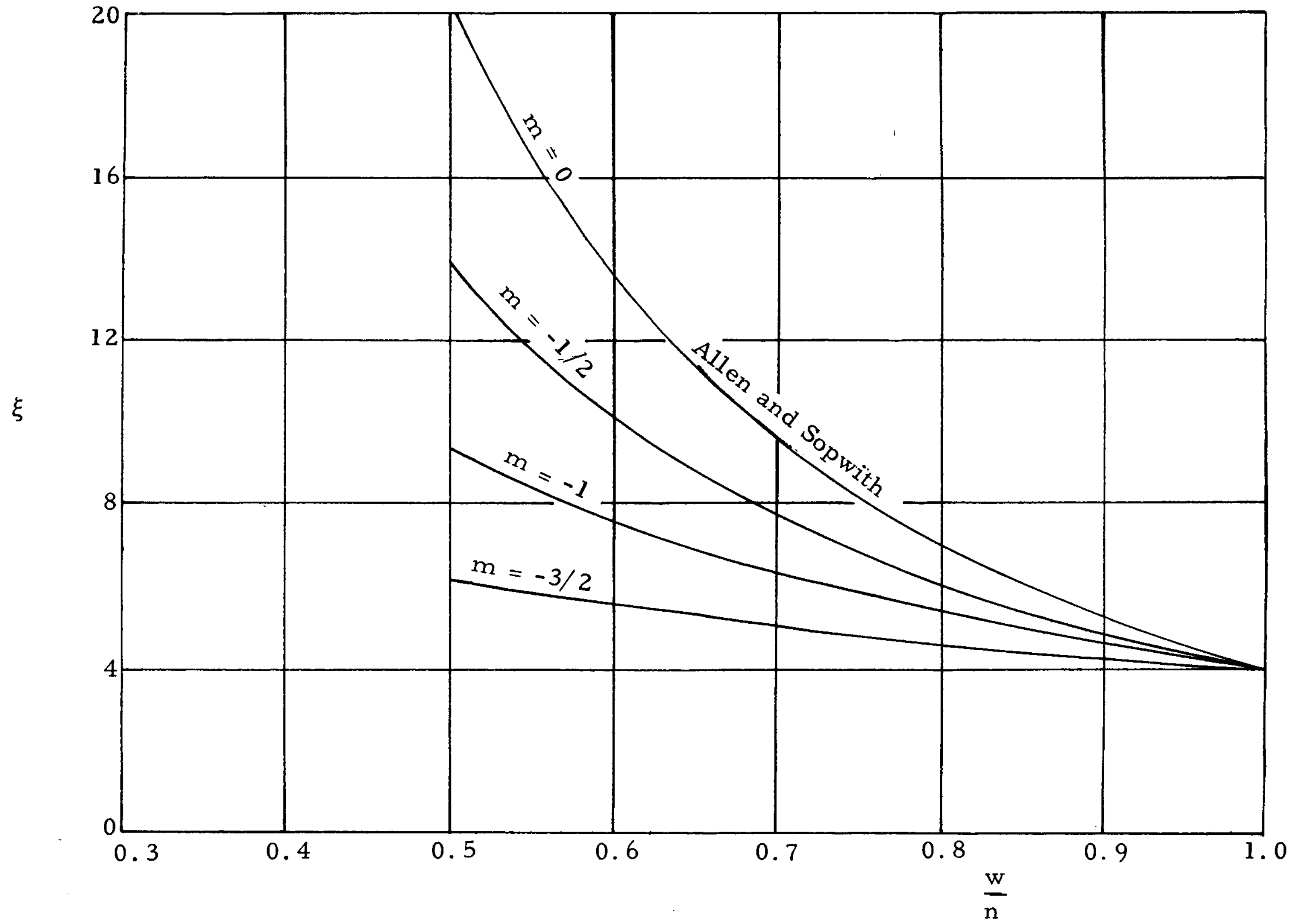

FIGURE 3. A DIMENSIONLESS PLOT SHOWING RELATIONSHIP BETWEEN ξ AND $\frac{w}{n}$ FOR VARIOUS VALUES OF m.

Once the exponent of strain-hardening of a cylinder material is selected by comparing its dimensionless stress-strain diagram with the curves in FIG 4, the stress and strain distributions across the cylinder radius can be readily determined from equations (13), (14), (19), (21) and (22). The stress distributions in cylinders made of four different materials ($m = 0, -1/2, -1, -3/2$) for a degree of yielding of $n = 2$, are plotted in FIG 5. The cylinder used in the illustrative example has an outside radius twice as large as its inner radius, in other words, $k = 2$.

A comparison between equation (19) and the third of equations (13) shows that the factor

$$\frac{Y - H}{H + 2Y} \quad \text{or} \quad \frac{\xi - 1}{1 + 2\xi}$$

has characteristics similar to that of the Poisson's ratio, ν. This factor assumes a value of 1/3 when $\xi = 4$, and approaches the value of 1/2 when $\xi \to \infty$.

CONCLUSIONS

Allen and Sopwith's solution for the thick-walled cylinder problem for an ideally plastic material was extended to strain-hardening material by introducing a strain-hardening function to the Tresca's criterion of plastic yielding. This strain-hardening function is assumed to have the form of a simple power function of the dimensionless radius in the plastic zone. The exponent of this simple power function may have different constant values, fractional or integral, depending on the strain-hardening property of the material concerned. As demonstrated in the foregoing analysis, the solution to this problem is in analytical and closed form if this exponential constant is equal to -3/2, -5/4, -1, -1/2 and 0. These exponents correspond to five materials with different strain-hardening characteristics similar to those observed in metals. Since the case of zero exponent coincides with that of Allen and Sopwith's ideally plastic material, their relative degree of strain-hardening can be compared by examining the curves in FIG 4, where the dimensionless quantities of maximum shearing stress and strain are plotted as ordinate and abscissa.

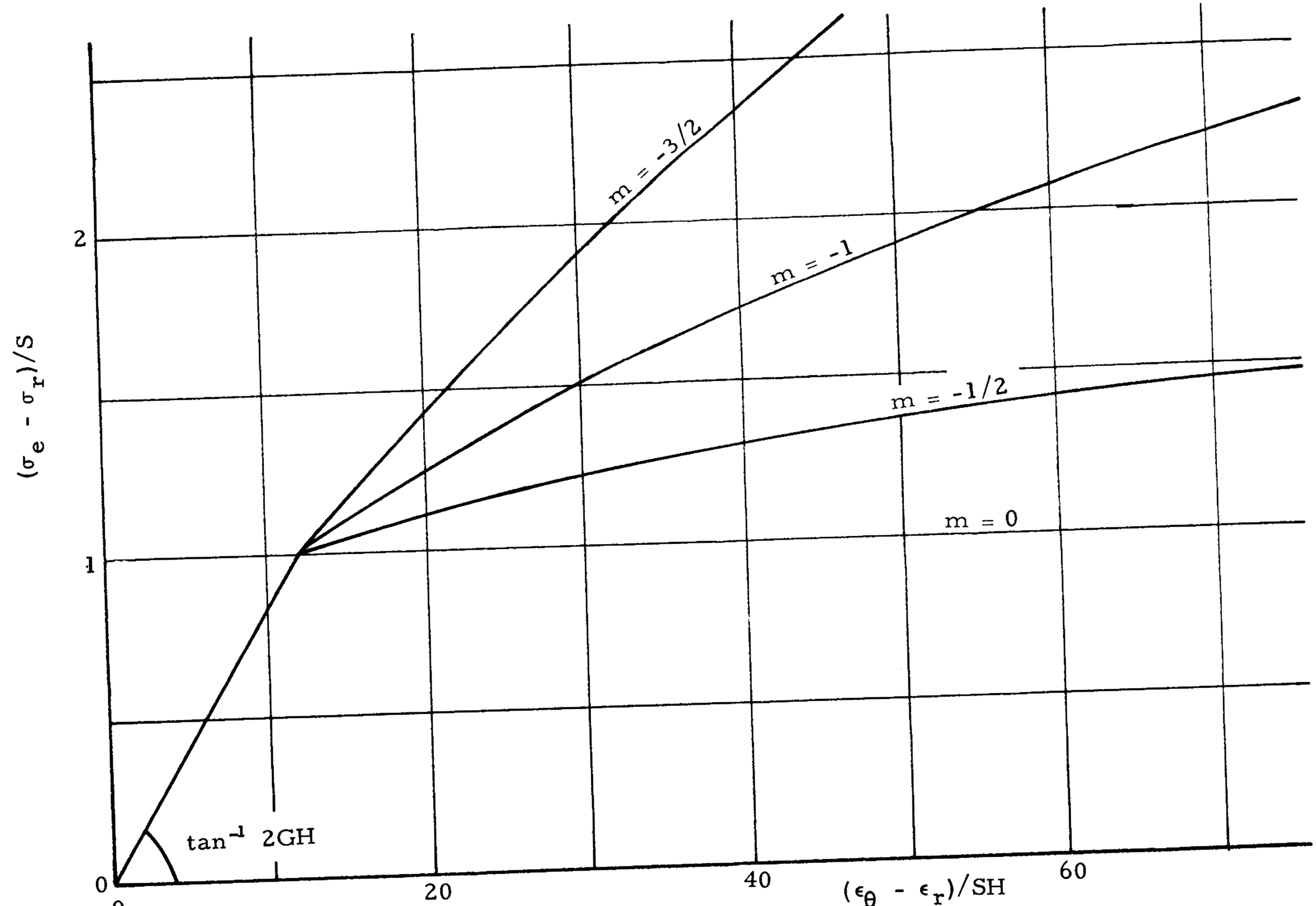

FIGURE 4. DIMENSIONLESS STRESS - STRAIN DIAGRAMS FOR FOUR DIFFERENT MATERIALS

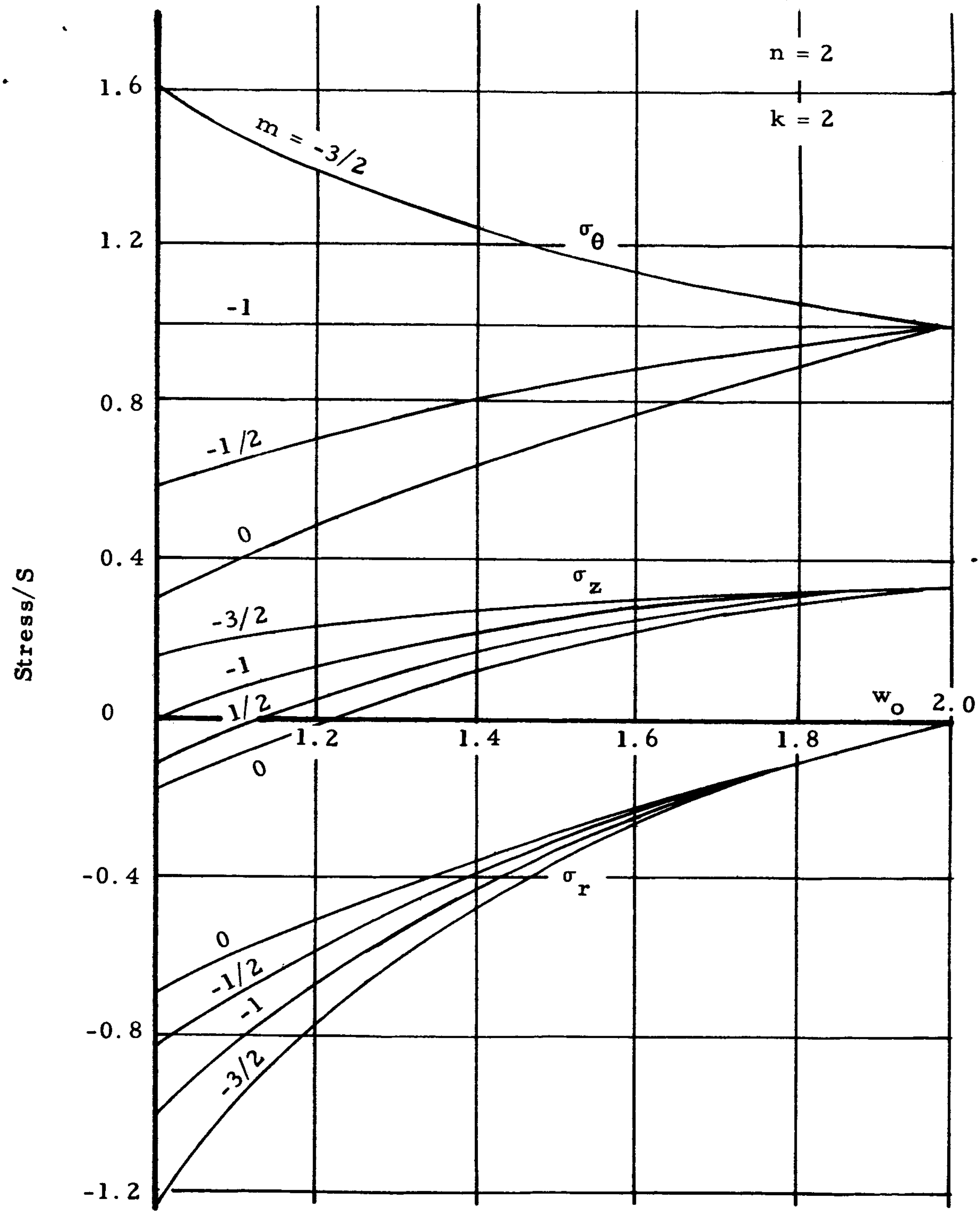

FIGURE 5. FULLY PLASTIC STRESS DISTRIBUTION IN AN INFINITE HOLLOW CYLINDER STRAINED BY INTERNAL PRESSURE

The assumption that the maximum shearing stress (effective stress) is proportional to a simple power function of the radius has been deduced from the solution of Allen and Sopwith as reasonable for engineering applications. Examination reveals that the value of this simple power function at any radius in the plastic zone is not appreciably affected by the amount of yielding that has penetrated beyond that redius. Within the realm of engineering significance and application, the example shown in this report (k = 2) seems to strengthen this assumption.

For exponents other than the five mentioned above, the integration involved may become very tedious for analytical solutions. In those cases, graphical or numerical processes may be used.

Although the cylinder used in the illustrative example was assumed to have clamped ends, this was mainly for the convenience of computation and is not a limitation of the theory.

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